

THE INDEX OF A NUMERICAL SEMIGROUP RING

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ABSTRACT. Let $R = k[[t^a, t^b, t^c]]$ be a complete intersection numerical semigroup ring over an infinite field k . The Löewy length, which is the Auslander index in this case, is computed in terms of the minimal generators of the semigroup: a, b and c . Examples provided show that the left hand side of Ding's inequality $\text{mult}(R) - \text{index}(R) - \text{codim}(R) + 1 \geq 0$ can be made arbitrarily large for rings R with $\text{edim}(R) = 3$. The index of a complete intersection numerical semigroup ring with embedding dimension greater than three is also computed.

INTRODUCTION

Let (R, \mathfrak{m}, k) be a local noetherian Gorenstein ring with maximal ideal \mathfrak{m} and residue field k and let M be a finitely generated R -module. The *Auslander's delta invariant* of the module M , denoted by $\delta(M)$, is the smallest non-negative integer μ such that there exists an exact sequence of R -modules, called Cohen-Macaulay approximation, $0 \rightarrow Y \rightarrow X \oplus R^\mu \rightarrow M \rightarrow 0$ such that X is a maximal Cohen-Macaulay module with no free direct summands and $\text{pd}_R Y < \infty$; see [1]. It is clear from the definition that $\delta(R^n) = n$ for every integer $n \geq 1$. If $\text{pd}_R M < \infty$, then $\delta(M)$ is the minimal numbers of generators of the module M . Moreover, a surjective homomorphism $M \rightarrow N \rightarrow 0$ induces an inequality between the delta invariants: $\delta(N) \leq \delta(M)$. In particular, $0 \leq \delta(R/\mathfrak{m}) \leq \delta(R/\mathfrak{m}^2) \leq \dots \leq \delta(R/\mathfrak{m}^i) \leq \delta(R/\mathfrak{m}^{i+1}) \leq \dots \leq 1$, for all $i \geq 1$.

The *index* of the ring R introduced by Auslander and studied by Ding in his thesis [5], denoted by $\text{index}(R)$, is defined as the minimum $i \geq 1$ such that $\delta(R/\mathfrak{m}^i) = 1$. The index is finite for all Gorenstein rings. Ding studies further in [5] the properties of the index over Gorenstein rings with infinite residue field. He proves that a ring R is regular if and only if $\text{index}(R)=1$. Moreover, he shows that the ring R is a hypersurface if and only if $\text{index}(R) = \text{mult}(R)$, where $\text{mult}(R)$ denotes the multiplicity of the ring R . Furthermore, if R is not regular, then

$$(*) \quad \text{mult}(R) - \text{index}(R) - \text{codim}(R) + 1 \geq 0.$$

In particular, $\text{mult}(R) \geq \text{index}(R)$. Here $\text{codim}(R) = \text{edim}(R) - \dim(R)$ denotes the codimension of the ring R .

Martsinkovsky [7] extends the notion of index for rings which are not necessarily noetherian, local or Gorenstein. He shows that the index is finite if R is a noetherian local ring. This index satisfies all of the properties mentioned above.

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This paper was motivated by the following:

Question. In the case when R is complete intersection ring is the left hand side of the inequality (*) bounded above by a constant?

We answer negatively this questions by providing examples which show that the left hand-side can be made arbitrarily large. Example 2.5 shows that for any integer n with $n \geq 2$, there exist complete intersection rings R_n with $\text{edim}(R_n) = 3$ and

$$\text{mult}(R_n) - \text{index}(R_n) - \text{codim}(R_n) + 1 = 2n - 3.$$

Proposition 3.8 shows that for any $n \in \mathbb{N}$ there exist complete intersection rings with $\text{edim}(R_n) = n$ such that

$$\text{mult}(R_n) - \text{index}(R_n) - \text{codim}(R_n) + 1 = 2^n - 2n.$$

These examples were found among the numerical semigroup rings.

Working with semigroup rings requires methods different from those used in defining the index originally. In Section 1 we introduce some notions that are needed in the paper.

The main result of the paper is Theorem 2.2; it leads to an explicit formula for the index of complete intersection numerical rings semigroup rings $R = k[[t^a, t^b, t^c]]$, where k is an infinite field and $a, b, c \in \mathbb{N}$ (with $\gcd(a, b, c) = 1$); see Remark 2.3. In Section 3 we make a discussion on the index in the case of any Gorenstein numerical semigroup ring and compute the index for some rings R with $\text{edim}(R) \geq 3$; see Proposition 3.8.

1. NUMERICAL SEMIGROUP RINGS

In this section we introduce some terminology that will be used in this paper. Let H be a semigroup generated by natural numbers $a_1 < a_2 < \dots < a_e$. The set $\mathbb{N} \setminus H$ is finite if and only if $\gcd(a_1, a_2, \dots, a_e) = 1$; in this case H is called a *numerical semigroup* and the *Fröbenius number* of H is given by

$$f(H) = \max\{h \notin H \mid h + i \in H, \text{ for every positive integer } i\}.$$

1.1. The associated *numerical semigroup ring* of the semigroup H is defined as

$$R = k[[H]] = k[[t^s \mid s \in H]]$$

and has the following properties: R is a local ring with maximal ideal $\mathfrak{m} = (t^{a_1}, \dots, t^{a_e})$, thus $\text{edim}(R) = e$; R is a domain, thus $\text{depth}(R) = 0$; $\dim(R) = 1$; $\text{mult}(R) = a_1$; and R is Gorenstein if and only if the semigroup H is symmetric (i.e. $s \in H$ if and only if $f(H) - s \notin H$).

A semigroup H is called *complete intersection* or *Gorenstein* if the corresponding semigroup ring $R = k[[H]]$ is complete intersection or Gorenstein respectively.

1.2. Let $R = k[[H]]$ be a Gorenstein numerical semigroup ring with k infinite field. The index of R is well defined and Watanabe shows in [5, Proposition 1.23] that

$$\text{index}(R) = \min\{i \mid \mathfrak{m}^i \subseteq (t^s), \text{ where } s \in H\}.$$

In particular, if we set $N_s = \min\{i \mid \mathfrak{m}^i \subseteq (t^s)\}$, for $s \in H$, then

$$\text{index}(R) = \min\{N_{a_j}\}_{j=1, \dots, e}.$$

If (R, \mathfrak{m}) is a noetherian local ring, then the *generalized Löwey length* is defined by

$$ll(R) = \min \{l | \mathfrak{m}^l \subseteq \mathbf{x}, \text{ for some system of parameters } \mathbf{x} \text{ of } R\}.$$

Watanabe's result shows that in the case when R is a Gorenstein numerical semigroup ring, the index of the ring is given by its generalized Löwey length.

Another notion that will be used later in the paper is the *order of an element* s of a semigroup H , $\text{ord}(s) = \max \left\{ \sum_{i=1}^e \alpha_i \mid s = \sum_{i=1}^e \alpha_i a_i \right\}$; see [2].

2. THE INDEX OF GORENSTEIN SEMIGROUP RINGS OF $\text{EDIM} = 3$

The normal semigroup rings of the form $R = k[[t^a, t^b]]$ are hypersurfaces and $\text{index}(R) = \text{mult}(R) = \min\{a, b\}$. Therefore, we turn our attention to the case when the semigroup ring is of embedding dimension three. These rings were studied by Herzog in [6] and by Watanabe in [9, Proposition 3].

2.1. Let H be a numerical semigroup minimally generated by three elements and set $R = k[[H]]$ for an infinite field k . The following are equivalent, after a possibly relabeling of the generators of H .

- (i) R is complete intersection;
- (ii) R is Gorenstein;
- (iii) There exists integers $p, x, y \geq 2$ such that $H = \langle a, b, c \rangle$ where

$$a \in \langle x, y \rangle \text{ with } a \notin \{x, y\} \text{ and } b = px, c = py, \gcd(x, y) = \gcd(a, p) = 1.$$

Moreover, when one (hence all) of these cases holds, $f(H) = pxy + pa - (a + b + c)$.

Theorem 2.2. *Let R be a complete intersection semigroup ring of embedding dimension three. With the notation from 2.1, set $a = a'x + a''y$ with a' and a'' non-negative integers. Then the following equalities hold.*

$$\begin{aligned} \text{(a) If } x < y, \text{ then } \min\{i | \mathfrak{m}^i \subseteq (t^a)\} &= \begin{cases} x + a' + y \frac{a''}{x} - 1, & \text{if } \frac{a''}{x} \in \mathbb{N}; \\ y + a' + a'' + (y - x) \left\lfloor \frac{a''}{x} \right\rfloor - 1, & \text{if } \frac{a''}{x} \notin \mathbb{N}. \end{cases} \\ \text{(b) } \min\{i | \mathfrak{m}^i \subseteq (t^b)\} &= \begin{cases} p + x - 1, & \text{if } a' \neq 0 \\ & \text{or } a' = 0, \text{ and } p < a''; \\ a'' + p \frac{x}{a''} - 1, & \text{if } a' = 0, a'' < p, \text{ and } \frac{x}{a''} \in \mathbb{N}; \\ p + x - 1 + (p - a'') \left\lfloor \frac{x}{a''} \right\rfloor, & \text{if } a' = 0, a'' < p, \text{ and } \frac{x}{a''} \notin \mathbb{N}. \end{cases} \end{aligned}$$

$$(c) \min\{i | \mathbf{m}^i \subseteq (t^c)\} = \begin{cases} p + y - 1, & \text{if } a'' \neq 0 \\ & \text{or } a'' = 0, \text{ and } p < a'; \\ a' + p \frac{y}{a'} - 1, & \text{if } a'' = 0, a' < p, \text{ and } \frac{y}{a'} \in \mathbb{N}; \\ p + y - 1 + (p - a') \left\lfloor \frac{y}{a'} \right\rfloor, & \text{if } a'' = 0, a' < p, \text{ and } \frac{y}{a'} \notin \mathbb{N}. \end{cases}$$

Proof. (a). Set $N = \min\{i | \mathbf{m}^i \subseteq (t^a)\}$. We may reduce to the case $a'' < x$ and prove that

$$N = \begin{cases} x + a' - 1, & \text{if } a'' = 0; \\ y + a' + a'' - 1, & \text{if } a'' \neq 0. \end{cases}$$

Indeed, assume that $a = a'x + a''y$ for some non-negative integers a' and a'' . There exist unique non-negative integers q and d'' such that $a'' = qx + d''$ with $0 \leq d'' < x$. If we set $d' = a' + qy$, then $a = d'x + d''y$. We have $d'' = 0$ if and only if $\frac{a''}{x} \in \mathbb{N}$, and then

$$x + d' - 1 = x + a' + y \frac{a''}{x} - 1.$$

We have $d'' \neq 0$ if and only if $\frac{a''}{x} \notin \mathbb{N}$, and then

$$y + d' + d'' - 1 = y + a' + a'' + (y - x) \left\lfloor \frac{a''}{x} \right\rfloor - 1.$$

For the rest of the proof we thus assume that $0 \leq a'' < x < y$. By definition, N is the minimum natural number with the property that for any non-negative integers u, v and w such that $u + v + w = N$ there exist non-negative integers α, β and γ with $\alpha \geq 1$ such that

$$ua + vb + wc = \alpha a + \beta b + \gamma c.$$

We may assume that $u = 0$, thus we get

$$vpx + wpy = \alpha a + \beta px + \gamma py.$$

Since $\gcd(a, p) = 1$, there exists an integer $\alpha' \geq 1$ such that $\alpha = p\alpha'$, so after dividing by p the equality above becomes

$$\begin{aligned} vx + wy &= \alpha'(a'x + a''y) + \beta x + \gamma y \iff \\ x(v - \alpha'a' - \beta) &= y(\gamma + \alpha'a'' - w). \end{aligned}$$

Since $\gcd(x, y) = 1$, there exists an integer z such that

$$\begin{aligned} v - \alpha'a' - \beta &= yz \\ \gamma + \alpha'a'' - w &= xz. \end{aligned}$$

Setting $v = N - \delta$ and $w = \delta$ for some $\delta \in \{0, 1, \dots, N\}$, and using that $\alpha' \geq 1$, $\beta \geq 0$ and $\gamma \geq 0$, we get that N is the minimum positive integer such that for each $0 \leq \delta \leq N$, there exists $z_\delta \in \mathbb{Z}$ such that

$$(2.2.1) \quad \frac{N - \delta - a'}{y} \geq z_\delta \geq \frac{-\delta + a''}{x}.$$

In particular, for $\delta = 0$ we have

$$(2.2.2) \quad \frac{N - a'}{y} \geq z_0 \geq \frac{a''}{x}.$$

Case $a'' = 0$.

The inequality (2.2.1) becomes

$$(2.2.3) \quad \frac{N - \delta - a'}{y} \geq z_\delta \geq -\frac{\delta}{x}.$$

If we assume that $N < x$, then by choosing $\delta = N$ the inequality above implies

$$0 > -\frac{a'}{y} \geq z_N \geq -\frac{N}{x} > -1,$$

which cannot happen for any $z_N \in \mathbb{Z}$. Thus, $N \geq x$.

If $0 \leq \delta < x$, then $z_\delta = 0$, thus $N \geq \delta + a'$. In particular, when $\delta = x - 1$ we get

$$(2.2.4) \quad N \geq x + a' - 1.$$

If $\delta \geq x$, write $\delta = qx + r$ with q, r non-negative integers with $0 \leq r < x$. Using (2.2.4) we obtain

$$\begin{aligned} \frac{N - a'}{y} + \delta \frac{y - x}{xy} &\geq \frac{x - 1}{y} + \delta \frac{y - x}{xy} \\ &\geq \frac{x - 1}{y} + \frac{y - x}{y} \\ &= 1 - \frac{1}{y} \\ &> 1 - \frac{1}{x} \\ &= \frac{x - 1}{x} \\ &\geq \frac{r}{x}. \end{aligned}$$

In particular, the inequality (2.2.3) holds if $z_\delta = -q$. Thus, we have $N = x + a' - 1$.

Case $a'' \neq 0$.

If we assume that $N < a''$, then by choosing $\delta = N$, the inequality (2.2.1) implies

$$0 \geq -\frac{a'}{y} \geq z_N \geq \frac{-N + a''}{x} > 0,$$

which is a contradiction. Thus, we must have $N \geq a''$. We apply inequality (2.2.1) to several cases of δ in order to show that we should have $N = a' + a'' + y - 1$.

If $\delta = a''$, then $z_\delta \geq 0$ and then $N \geq a' + a''$.

If $0 \leq \delta < a''$, then $z_\delta \geq 1$ and then $N \geq a' + y + \delta$. In particular, for $\delta = a'' - 1$,

$$N \geq a' + a'' + y - 1.$$

If $\delta > a''$, then write $\delta - a'' = qx + r$ for non-negative integers q, r and $0 \leq r < x$. Using that $N \geq a' + a'' + y - 1$ and $\delta \geq a'' + 1$, we obtain

$$\begin{aligned}
\frac{N - a'}{y} + \delta \frac{y - x}{xy} &\geq \frac{a'' + y - 1}{y} + (a'' + 1) \left(\frac{1}{x} - \frac{1}{y} \right) \\
&= 1 + \frac{1}{x} - \frac{2}{y} + \frac{a''}{x} \\
&= \frac{x - 1}{x} + 2 \left(\frac{1}{x} - \frac{1}{y} \right) + \frac{a''}{x} \\
&> \frac{r}{x} + \frac{a''}{x}.
\end{aligned}$$

Then the inequality (2.2.1) holds by taking $z_\delta = -q$. Therefore, $N = a' + a'' + y - 1$.

(b). *Case $a' \neq 0$.*

First, we show that $\mathfrak{m}^{p+x-1} \subseteq (t^b)$. This is equivalent with showing that for any u, v and w non-negative integers such that $u + v + w = p + x - 1$ there exist non-negative integers α, β and γ with $\beta \neq 0$ such that

$$ua + vb + wc = \alpha a + \beta b + \gamma c.$$

If $v \neq 0$, then this is clear. Assume that $v = 0$, $u = p + x - 1 - \delta$ and $w = \delta$ where $\delta \in \{0, \dots, p + x - 1\}$. We consider the two cases: $\delta \leq x - 1$ and $\delta > x - 1$.

In the case $\delta \leq x - 1$, we have:

$$\begin{aligned}
ua + wc &= (p + x - 1 - \delta)a + \delta py \\
&= (x - 1 - \delta)a + a'px + (\delta + a'')py \\
&= (x - 1 - \delta)a + a'b + (\delta + a'')c.
\end{aligned}$$

Since $a' \neq 0$, we have written $ua + wc$ in the desired format.

In the case $\delta > x - 1$, we set $\theta = \delta - x + 1 > 0$. Remark that $p - \theta \geq 0$. We have:

$$\begin{aligned}
ua + wc &= (p + x - 1 - \delta)a + \delta py \\
&= (p - \theta)a + pxy + (\theta - 1)py \\
&= (p - \theta)a + yb + (\theta - 1)c.
\end{aligned}$$

Next, we show that $\mathfrak{m}^{p+x-2} \not\subseteq (t^b)$. Assume that there exist non-negative integers α, β and γ with $\beta \neq 0$ and $(p - 1)a + (x - 1)c = \alpha a + \beta b + \gamma c$. This is equivalent to

$$(2.2.5) \quad pa + (x - 1)py = (\alpha + 1)a + \beta px + \gamma py.$$

Since $\gcd(a, p) = 1$, there exists a positive integer α' such that $\alpha + 1 = p\alpha'$. Thus, the last equality above is equivalent to

$$\begin{aligned}
a'x + a''y + (x - 1)y &= \alpha'(a'x + a''y) + \beta x + \gamma y \iff \\
y(a'' - 1 + x - \alpha'a'' - \gamma) &= x(\alpha'a' + \beta - a') \iff \\
y[x - a''(\alpha' - 1) - \gamma] &= x[a'(\alpha' - 1) + \beta].
\end{aligned}$$

Since $\gcd(x, y) = 1$ and $a'(\alpha' - 1) + \beta > 0$, there exists a positive integer z such that

$$x - a''(\alpha' - 1) - \gamma = xz.$$

This together with the fact that $a''(\alpha' - 1) + \gamma \geq 0$, implies that $z = 1$ thus

$$a''(\alpha' - 1) = 0 \text{ and } \gamma = 0.$$

If $a'' \neq 0$, then $\alpha' = 1$ and, after dividing by p , the equality (2.2.5) becomes $(x-1)y = \beta x$, which is a contradiction since $\gcd(x, y) = 1$. If $a'' = 0$, then $a = a'x$ and, after dividing by p , the equality (2.2.5) becomes $(x-1)y = (\alpha' - 1)a'x + \beta x$, which is again a contradiction.

Case $a' = 0$. In particular, it implies that $a'' \neq 0$.

If $p < a''$, then apply part (a) with $a, b, c, p, x, y, a', a''$ taken to be $px, py, a''y, y, p, a'', x$ and respectively 0.

If $p > a''$, then apply part (a) with $a, b, c, p, x, y, a', a''$ taken to be $px, a''y, py, y, a'', p, 0$, and respectively x .

(c) follows from (b) due to the symmetry of the statement in x and y . \square

Remark 2.3. Theorem 2.2 and the results from 1.2 allow us now to give a “formula” for the index of a numerical semigroup ring of embedding dimension three in terms of the generators of the semigroup. Indeed, $\text{index}(R) = \min\{N_a, N_b, N_c\}$, and N_a, N_b and N_c were computed in Theorem 2.2. We give below some special cases.

Corollary 2.4. *Let R be a complete intersection semigroup ring of embedding dimension three. With the notation from 2.1, set $a = a'x + a''y$ with a' and a'' non-negative integers and assume that $a'' < x < y$.*

(a) *If $a' \neq 0$ and $a'' \neq 0$, then*

$$\text{index}(R) = \min\{y + a' + a'' - 1, p + x - 1\}.$$

(b) *If $a' = 0$ and $p < a''$, then*

$$\text{index}(R) = \min\{y + a'' - 1, p + x - 1\}.$$

(c) *If $a'' = 0$, then*

$$\text{index}(R) = \min\{p + x - 1, x + a' - 1\}.$$

Example 2.5. Let $R = k[[t^{4n}, t^{(4n+1)(2n-1)}, t^{(4n+1)(2n+1)}]]$ where k is an infinite field and $n \geq 2$. Then R is a complete intersection ring with

$$\text{mult}(R) - \text{index}(R) - \text{codim}(R) + 1 = 2n - 3.$$

Proof. If we let $a = 4n$, $x = 2n - 1$, $y = 2n + 1$ and $p = 4n + 1$, we obtain by Corollary 2.4(a) that $\text{index}(R) = \min\{2n + 2, 6n - 1\} = 2n + 2$. So, using now 1.1 we get $\text{mult}(R) - \text{index}(R) - \text{codim}(R) + 1 = 4n - (2n + 2) - 2 + 1 = 2n - 3$. \square

Another way of expressing the index in terms of the generators of the semigroup is to use the Fröbenius number and the order function defined in Section 1.

Proposition 2.6. *Let R be a complete intersection numerical semigroup ring of embedding dimension three. With the notations from 1.2 the following equalities hold.*

- (a) $N_a = \text{ord}(f(H) + a) + 1$.
- (b) $N_b = \text{ord}(f(H) + b) + 1$.
- (c) $N_c = \text{ord}(f(H) + c) + 1$.

Proof. The Fröbenius number is given by $f(H) = pxy + pa - (a + b + c)$, see 2.1.

(a). By definition $\text{ord}(n) = \max\{\alpha + \beta + \gamma \mid n = \alpha a + \beta b + \gamma c\}$. Let α, β and γ such that $f(H) + a = \alpha a + \beta b + \gamma c$. Thus,

$$pxy + pa = \alpha a + (\beta + 1)b + (\gamma + 1)c.$$

Since $\gcd(a, p)=1$, there exists a non-negative integer α' such that $\alpha = p\alpha'$, so the equality above becomes

$$(2.6.1) \quad xy + a = \alpha' a + (\beta + 1)x + (\gamma + 1)y.$$

If $\alpha' > 0$, then we have

$$\begin{aligned} xy &= (\alpha' - 1)(a'x + a''y) + (\beta + 1)x + (\gamma + 1)y \iff \\ x[y - (\alpha' - 1)a' - (\beta + 1)] &= y[a''(\alpha' - 1) + (\gamma + 1)]. \end{aligned}$$

Since $\gcd(x, y) = 1$ it follows that y divides $y - (\alpha' - 1)a' - (\beta + 1)$, which is in contradiction with the fact that $0 < y - (\alpha' - 1)a' - (\beta + 1) < y$.

Thus $\alpha' = 0$, from where $\alpha = 0$ and the equality (2.6.1) becomes

$$\begin{aligned} xy + a'x + a''y &= (\beta + 1)x + (\gamma + 1)y \iff \\ x[y + a' - (\beta + 1)] &= y[(\gamma + 1) - a'']. \end{aligned}$$

Since $\gcd(x, y) = 1$, there exists $z \in \mathbb{Z}$ such that

$$\begin{aligned} y + a' - (\beta + 1) &= yz \\ (\gamma + 1) - a'' &= xz. \end{aligned}$$

Using in addition that $\beta + 1 > 0$ and $\gamma + 1 > 0$ we get that the $\text{ord}(f(H) + a)$ is

$$\max \left\{ y + a' + a'' + z(x - y) - 2 \mid \exists z \in \mathbb{Z} \text{ such that } \frac{a' - 1}{y} + 1 \geq z \geq -\frac{a'' - 1}{x} \right\}.$$

The function $\varphi(z) = y + a' + a'' + z(x - y) - 2$ is decreasing as $\varphi'(z) = x - y < 0$. Therefore, the maximum of the function φ , is attained when z is minimum, under the given conditions. If $a'' = 0$, then $z_{\min} = 1$, and thus

$$\text{ord}(f(H) + a) = y + a' + a'' + (x - y) - 2 = x + a' - 2.$$

Assume that $a'' \geq 1$ and let $a'' = x \left\lfloor \frac{a''}{x} \right\rfloor + r$, where $0 \leq r < x$. If $r = 0$, then

$z_{\min} = -\frac{a''}{x} + 1$. Therefore,

$$\begin{aligned} \text{ord}(f(H) + a) &= y + a' + a'' + \frac{a''}{x}(y - x) + x - y - 2 \\ &= x + a' + y \frac{a''}{x} - 2. \end{aligned}$$

If $r \neq 0$, then $z_{\min} = -\left\lfloor \frac{a''}{x} \right\rfloor$. Therefore,

$$\text{ord}(f(H) + a) = y + a' + a'' + \left\lfloor \frac{a''}{x} \right\rfloor (y - x) - 2.$$

Using Theorem 2.2(a) we get the desired equality.

(b). Let α, β and γ non-negative integers such that $f(H) + b = \alpha a + \beta b + \gamma c$. Thus,

$$(2.6.2) \quad pxy + pa = (\alpha + 1)a + \beta b + (\gamma + 1)c.$$

Since $\gcd(a, p) = 1$, there exists a positive integer α' such that $\alpha + 1 = p\alpha'$, thus dividing (2.6.2) by p we get

$$\begin{aligned} xy + a &= \alpha'a + \beta x + (\gamma + 1)y \iff \\ xy &= (\alpha' - 1)(a'x + a''y) + \beta x + (\gamma + 1)y \iff \\ x[y - (\alpha' - 1)a' - \beta] &= y[(\alpha' - 1)a'' + (\gamma + 1)] \neq 0. \end{aligned}$$

Since $\gcd(x, y) = 1$ it follows that y divides the positive number $y - (\alpha' - 1)a' - \beta$, which is less or equal to y . Thus,

$$\begin{aligned} \beta &= 0 \\ (\alpha' - 1)a' &= 0 \\ (\alpha' - 1)a'' + (\gamma + 1) &= x. \end{aligned}$$

If $a' \neq 0$, then $\alpha' = 1$ and thus $\alpha = p - 1$ and $\gamma = x - 1$. This implies that

$$\text{ord}(f(H) + b) = p + x - 2.$$

If $a' = 0$, then $a = a''y$ and (2.6.2) becomes

$$\begin{aligned} pxy + pa''y &= (\alpha + 1)a''y + (\gamma + 1)py \iff \\ px + pa'' &= (\alpha + 1)a'' + (\gamma + 1)p \iff \\ a''(p - \alpha - 1) &= p(\gamma + 1 - x). \end{aligned}$$

Since $\gcd(a'', p) = 1$, there exists an integer z such that

$$\begin{aligned} p - \alpha - 1 &= pz \\ \gamma + 1 - x &= a''z. \end{aligned}$$

Using in addition that $\alpha + 1 > 0$ and $\gamma + 1 > 0$ we get that $\text{ord}(f(H) + b)$ is

$$\max \left\{ (p + x - 2) + (a'' - p)z \mid \exists z \in \mathbb{Z} \text{ such that } 0 \geq z \geq \frac{1 - x}{a''} \right\}.$$

Set $\varphi(z) = (p + x - 2) + (a'' - p)z$, so $\varphi'(z) = a'' - p$. If $p < a''$, then φ is an increasing function whose maximum is attained when z is maximum, that is $z_{\max} = 0$. Thus,

$$\text{ord}(f(H) + b) = p + x - 2.$$

If $p > a''$, then φ is a decreasing function whose maximum is attained when z is minimum. Let $x = a'' \left\lfloor \frac{x}{a''} \right\rfloor + r$, where $0 \leq r < a''$. If $r = 0$, then $z_{\min} = -\frac{x}{a''} + 1$. Therefore,

$$\text{ord}(f(H) + b) = a'' + p \frac{x}{a''} - 2.$$

If $r \neq 0$, then $z = -\left\lfloor \frac{x}{a''} \right\rfloor$. Therefore,

$$\text{ord}(f(H) + b) = p + x - 2 + (p - a'') \left\lfloor \frac{x}{a''} \right\rfloor.$$

Using Theorem 2.2(b) we get the desired equality.

(c) follows from part (b), due to the symmetry of the statement in x and y . \square

3. THE INDEX OF A COMPLETE INTERSECTION SEMIGROUP RING OF EDIM > 3

In this section let $H = \langle a_1, a_2, \dots, a_e \rangle$ be a semigroup with $\gcd(a_1, \dots, a_e) = 1$ and $a_1 < a_2 < \dots < a_e$ and set $R = k[[H]]$ for an infinite field k .

If R is Gorenstein, then $\min\{i | \mathfrak{m}^i \subseteq (t^{a_1})\} = \text{ord}(f(H) + a_1) + 1$; see [8, Lemma 2.5]. Proposition 2.6 proves that this is true for all $i = 1, \dots, e$ when $e = 3$. Yi-Huang Shen and Lance Bryant confirmed [3] that a more general result holds. Although the proof they suggest uses techniques not used in this paper, we include it here as it does not appear in literature. For details on terminology and background results see [2].

Proposition 3.1. *If $R = k[[H]]$ is a Gorenstein numerical semigroup ring, then*

$$\min\{i | \mathfrak{m}^i \subseteq (t^s)\} = \text{ord}(f(H) + s) + 1,$$

for all $s \in H \setminus \{0\}$.

Proof. We recall, see [2], that the Apéry set of $n \in H \setminus \{0\}$ is defined by $\text{Ap}(H; n) = \{w \in H \mid w - n \notin H\}$. When $s \in H$, then

$$\text{Ap}(H; s) = \{w_0, w_1, \dots, w_{s-1}\},$$

where $0 = w_0 < w_1 < \dots < w_{s-1} = f(H) + s$.

A homogeneous element $t^w \in \mathfrak{m}^i$ belongs to (t^s) if and only if $t^w = t^s \cdot t^u$ for some $u \in H$ if and only if $w - s \in H$. On the other hand, $t^w \in \mathfrak{m}^i$ if and only if $\text{ord}(w) \geq i$. Therefore, we get the first equality of:

$$\begin{aligned} N_s &= \min\{i \mid \text{for all } w \in H \text{ such that } \text{ord}(w) \geq i, \text{ we have } w \notin \text{Ap}(H; s)\} \\ &= \max\{\text{ord}(w) \mid w \in \text{Ap}(H; s) \setminus \{0\}\} + 1. \end{aligned}$$

The second equality follows from the definition of the Apéry set. It is easy to check that the proof of [2, Proposition 3.6] holds also in when a_1 is replaced by any $s \in H$. Thus, for a symmetric semigroup H we obtain

$$w_i + w_j = w_{s-1} = f(H) + s \text{ for all } i + j = s - 1.$$

In particular,

$$\text{ord}(w) \leq \text{ord}(f(H) + s), \text{ for all } w \in \text{Ap}(H; s) \setminus \{0\}.$$

The desired conclusion now follows. \square

Corollary 3.2. *If $R = k[[H]]$ is a Gorenstein numerical semigroup ring, then*

$$\text{index}(R) = \min_{i=1, \dots, e} \{\text{ord}(f(H) + a_i)\} + 1.$$

Remark 3.3. Although Proposition 2.6 follows now from Proposition 3.1, we keep our proof in the embedding dimension three case to show that the complexity of computations does not decrease when it comes to get a precise formula for the index in terms of the generators of the semigroup.

The next example shows that Proposition 2.6 does not hold in the case when R is not a Gorenstein ring.

Example 3.4. Let k be a field and $R = k[[t^4, t^5, t^{11}]]$. If f is the Fröbenius number of the semigroup $\langle 4, 5, 11 \rangle$, then

$$N_4 \neq \text{ord}(f + 4) + 1,$$

Indeed, $f = 7$ and $\text{ord}(f + 4) + 1 = \text{ord}(11) + 1 = 2$ and $N_4 \neq 2$ as $t^{10} \in \mathfrak{m}^2 \setminus (t^4)$.

In [9, Lemma 1], K. Watanabe shows how one can construct complete intersection numerical semigroup rings. His result is generalized by Delorme [4, Proposition 10].

3.5. If H is a complete intersection semigroup generated by the natural numbers $a_1 < a_2 < \dots < a_e$ with $\gcd(a_1, a_2, \dots, a_e) = 1$, then $H' = \langle a, pH \rangle$ is a complete intersection for all a and p such that $a = \sum_{i=1}^e \alpha_i a_i$ with $\sum_{i=1}^e \alpha_i > 1$ and $\gcd(a, p) = 1$. Moreover, in this case we have $f(H') = p \cdot f(H) + (p - 1)a$.

3.6. [9] Set $H_{n,a} = \langle 2^n, 2^n + a, \dots, 2^n + 2^i a, \dots, 2^n + 2^{n-1} a \rangle$ with $n \geq 1$ and a is a positive odd integer. The semigroup $H_{n,a}$ is a complete intersection as it is obtained inductively: $H_{n,a} = \langle 2^n + a, 2H_{n-1,a} \rangle$, for all $n \geq 2$.

Lemma 3.7. *The semigroup $H_{n,a}$ defined in 3.6 has the Fröbenius number*

$$f(H_{n,a}) = (n - 1)2^n + (2^n - 1)a.$$

Proof. It follows by induction on n using 3.5. \square

Proposition 3.8. *Let $R_{n,a} = k[[H_{n,a}]]$ be a semigroup ring with $H_{n,a}$ given in 3.6 and k an infinite field. Then*

$$\text{index}(R_{n,a}) = n + 1.$$

In particular,

$$\text{mult}(R_{n,a}) - \text{index}(R_{n,a}) - \text{codim}(R_{n,a}) + 1 = 2^n - 2n.$$

Proof. Using Corollary 3.2 it is enough to compute the minimum of $\text{ord}(f(H_{n,a}) + 2^n)$ and $\text{ord}(f(H_{n,a}) + 2^n + 2^k a)$ for $k = 0, \dots, n - 1$.

Claim. $\text{ord}(f(H_{n,a}) + 2^n) = n$. Indeed, by Lemma 3.7 we have

$$\begin{aligned} f(H_{n,a}) + 2^n &= n2^n + (2^n - 1)a \\ &= (2^n + a) + (2^n + 2a) + \dots + (2^n + 2^{n-1}a). \end{aligned}$$

Thus, $\text{ord}(f(H_{n,a}) + 2^n) \geq n$. In general, we assume that

$$f(H_{n,a}) + 2^n = \alpha_0 2^n + \alpha_1 (2^n + a) + \alpha_2 (2^n + 2a) + \dots + \alpha_n (2^n + 2^{n-1}a).$$

This is equivalent to

$$(2^n + a) + (2^n + 2a) + \dots + (2^n + 2^{n-1}a) = \alpha_0 2^n + \alpha_1 (2^n + a) + \alpha_2 (2^n + 2a) + \dots + \alpha_n (2^n + 2^{n-1}a).$$

Thus, we have

$$2^n \cdot [n - (\alpha_0 + \alpha_1 + \dots + \alpha_n)] = a \cdot (\alpha_1 + 2\alpha_2 + \dots + 2^{n-1}\alpha_n - 2^n + 1),$$

which implies, since a is odd, that there exists an integer z such that

$$\begin{aligned} n - (\alpha_0 + \alpha_1 + \dots + \alpha_n) &= az, \text{ and} \\ \alpha_1 + 2\alpha_2 + \dots + 2^{n-1}\alpha_n - 2^n + 1 &= 2^n z. \end{aligned}$$

In particular, the second equality gives that $z \geq 0$ and the first that

$$\alpha_0 + \alpha_1 + \dots + \alpha_n = n - az \leq n.$$

Since the order is given by the maximum of such sums, the claim holds.

Next, we show that $\text{ord}(f(H_{n,a}) + 2^n + 2^k a) \geq n$ for all $k = 0, \dots, n - 1$, concluding our proof.

If $k = 0$, then

$$\begin{aligned} f(H_{n,a}) + (2^n + a) &= n2^n + (2^n - 1)a + a \\ &= (n + 1)2^n. \end{aligned}$$

Thus, $\text{ord}(f(H_{n,a}) + 2^n + a) \geq n + 1$.

If $1 \leq k \leq n - 2$, then

$$\begin{aligned} f(H_{n,a}) + (2^n + 2^k a) &= n2^n + (2^n - 1)a + 2^k a \\ &= (2^n + a) + \cdots + (2^n + 2^{k-1} a) \\ &\quad + 2(2^n + 2^{k+1} a) + \cdots + (2^n + 2^{n-1} a). \end{aligned}$$

Thus, $\text{ord}(f(H_{n,a}) + 2^n + 2^k a) \geq n$.

If $k = n - 1$, then

$$\begin{aligned} f(H_{n,a}) + (2^n + 2^{n-1} a) &= n2^n + (2^n - 1)a + 2^{n-1} a \\ &= (2^n + a) + \cdots + (2^n + 2^{k-1} a) + \cdots + (2^n + 2^{n-2} a) + (1 + a)2^n. \end{aligned}$$

Thus, $\text{ord}(f(H_{n,a}) + 2^n + 2^{n-1} a) \geq n + a$. Therefore, the minimum order is n , so $\text{index}(R_{n,a}) = n + 1$. The last equality follows from 1.1 \square

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